3 Partial Differential Equations

3.1 Introduction

Have solved ordinary differential equations, *i.e.* ones where there is one independent and one dependent variable. Only ordinary differentiation is therefore involved. As the world is three-dimensional, most differential equations are functions of three spatial variables, eg $(x\,,\,y\,,\,z)$, and maybe time t also. Typical example is Laplace equation

$$\nabla^2\,V(\underline{r}) = 0\,,$$

where $V(\underline{r})$ is the electrostatic potential in region where there is no charge. The operator ∇^2, called the Laplacian, was introduced last year. In Cartesian coordinates

$$\nabla^2\,V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\,. \tag{81}$$

Another important example is the time-independent Schrödinger equation for 1 particle

$$-\frac{\hbar^2}{2m}\,\nabla^2\,\Psi(\underline{r}) + V(\underline{r})\,\Psi(\underline{r}) = E\,\Psi(\underline{r})\,, \tag{82}$$

for the quantum-mechanical motion of a particle of mass m in a potential $V(\underline{r})$. $\Psi(\underline{r})$ is the particle's wave function and $\hbar = h/2\pi$, where h is Planck's constant. There are many more examples that you will come across later in your degree programme.

3.2 Classification of Differential Equations

Before considering various differential equations (DE) in detail it is worth defining some of the terms used to classify these equations into different types. The following terms are used:

Order. The order of a DE is the order of its highest derivative, so

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots a_0(x)y = 0 \tag{83}$$

is a DE of order n. This definition holds even if there are several variables, so

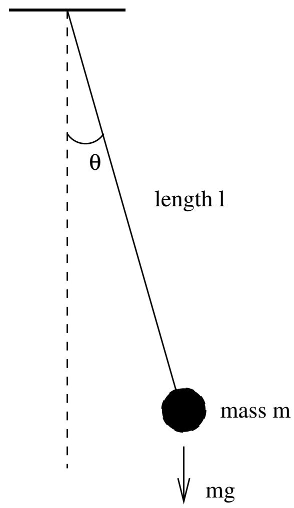
$$\frac{\partial^3 y}{\partial x^3} + \frac{\partial^2 y}{\partial t^2} = 0\,, \tag{84}$$

is a third-order.

Linearity. A linear DE can be written entirely as a linear function. *i.e.* no powers above the first power, of the unknown function and its derivative. So

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots a_0(x)y = b(x) \tag{85}$$

is linear if the a_i's and b are functions of x only. It is non-linear if any of the a_i's depend on y. Example: a pendulum



$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \tag{86}$$

is non-linear in θ. However if θ is small then $\sin\theta \approx \theta$ and the DE

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{87}$$

is linear. Linear DE's are important because they are easier to solve.

Ordinary/Partial. If an unknown function, eg y, is a function of only one variable, eg x, then one gets ordinary DEs such as

$$\frac{dy}{dx} = c\ . \tag{88}$$

If y is function of more than one variable, eg x and t then one gets a partial DE eg

$$\frac{\partial y(x,t)}{\partial x} + \frac{\partial y(x,t)}{\partial t} = c \tag{89}$$

provided the variables, x and t, are independent. If the variables are dependent, eg $x = f(s,t)$, then it is necessary to specify which are held constant

$$\left.\frac{\partial y(x,t)}{\partial x}\right|_t = c(s,t) \tag{90}$$

Such constructions are familiar from thermodynamics where P (pressure), V (volume) and T (temperature) are all inter-related eg by the ideal gas equation $PV = nRT$ and many functions, such as entropy S, have to written as partial DEs. This means that

$$\left.\frac{\partial S}{\partial T}\right|_P \neq \left.\frac{\partial S}{\partial T}\right|_V \tag{91}$$

Homogeneous. Means slightly different things for linear and non-linear DEs. Will only consider the linear DE case.

A matrix equation such as $\underline{Ax} = \underline{b}$ is homogeneous if $\underline{b} = 0$. Similarly, a (second-order) DE

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \tag{92}$$

is homogeneous if $G(x) = 0$ and is inhomogenous if $G(x) \neq 0$. Solving the homogeneous DE is usually the first step in solving an inhomogenous DE. We will restrict ourselves to homogenous DEs.

Solutions. By a solution of an ordinary DE

$$F(x, y(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots) = 0$$

we mean some function $y = u(x)$ in the range $a < x < b$ for which the problem is defined. This solution can always be verfied by direct substitution. Does

$$F(x, u(x), \frac{du}{dx}, \frac{d^2u}{dx^2}, \ldots) = 0 \ ?$$

Uniqueness. A DE in general will have more than one solution because:

1. There are unknown constants which can only be determined by the boundary conditions. Boundary conditions give information about the unknown function (or its derivatives) at some point. Eg $y = 0$ at $x = 1$ is a boundary condition. n boundary conditions are required to determine constants for an n^{th}-order equation. So a second-order DE requires 2 boundary conditions.

2. For an n^{th}-order DE there are usually n functions, $u(x)$, satisfying the DE. So a second-order DE has 2 solutions. Which solution is correct is often determined by the physics of the problem.

Existence. There is no guarantee that a DE will have a solution of the form $u(x)$.

Superposition Principle

If V_1 and V_2 are two solutions of any <u>linear</u>, <u>homogeneous</u> DE such as $\nabla^2 V(\underline{r}) = 0$, then $V = c_1 V_1 + c_2 V_2$, where c_1 and c_2 are arbitrary constants, is another solution. Used extensively for ordinary DEs, eg simple harmonic motion problem; is equally valid for partial DEs. This ability to add solutions is called the <u>Superposition Principle</u>. Of fundamental importance in Quantum Mechanics. Will exploit the superposition principle extensively when solving partial DEs.

3.3 Separation of variables

Most DEs that characterise physical problems depend on many variables and cannot be directly solved.Sometimes can solve these multi-dimensional problems by separation of variables which turns a partial DE in n variables into n ordinary DEs each in one variable. Take an $n = 2$ example

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial y^2} = 0\,. \tag{93}$$

If this is separable we can write $u(x,y) = X(x)Y(y)$ which gives

$$a(x,y)Y(y)\frac{d^2X}{dx^2} + b(x,y)X(x)\frac{d^2Y}{dy^2} = 0\,, \tag{94}$$

or, dividing through by XY and re-arranging:

$$\frac{a(x,y)}{X(x)}\frac{d^2X}{dx^2} = -\frac{b(x,y)}{Y(y)}\frac{d^2Y}{dy^2}\,. \tag{95}$$

This equation is separable *provided* that the left-hand side can be written totally in terms of x and the right-hand side totally in terms of y. This may require some re-arrangement between $a(x,y)$ and $b(x,y)$ to give $A(x)$ and $B(y)$, respectively functions of x and y only.

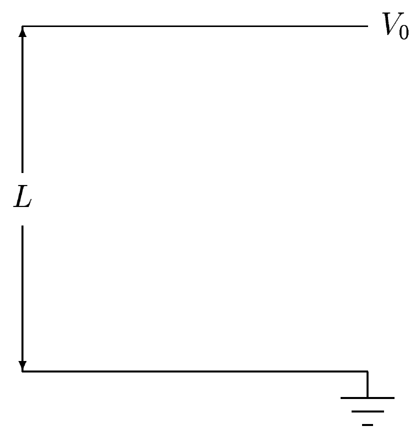
If eq. is separable, then have relationship of form $f(x) = g(y)$. Since relationship holds for all values of x and y, must mean that $f(x) = c = g(y)$, where c is some constant, often for convenience written as a square eg l^2. Can solve separately two equations

$$\frac{A(x)}{X(x)}\frac{d^2X}{dx^2} = c, \qquad \frac{B(y)}{Y(y)}\frac{d^2Y}{dy^2} = -c\,. \tag{96}$$

Note that separability depends on the coordinates chosen, it may be necessary to change coordinates.

Laplace's equation in Cartesian coordinates

Let us illustrate this with a physical example. Consider two infinitely large conducting plates. The one at $z = 0$ is earthed while that at $z = L$ is kept at a constant voltage V_0.



What is the potential between the two plates? You all know that the answer must be $V = V_0 z/L$ but we are going to derive this by solving the partial differential equation. This will demonstrate the techniques to be used in more complex cases.

Between the two plates, there is no charge and so the potential in this region satisfies Laplace's equation

$$\nabla^2 V(\underline{r}) = 0\,.$$

The boundary conditions to be applied are that, independent of the values of x and y, on the plates

$$\begin{aligned} V &= 0 \quad \text{at} \quad z = 0\,, \\ V &= V_0 \quad \text{at} \quad z = L\,. \end{aligned} \tag{97}$$

Since the boundary conditions are expressed easily in terms of Cartesian coordinates, it makes obvious sense to attack the problem in this coordinate system. [Could also use cylindrical polar coordinates.] In this system, Laplace's equation becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0\,.$$

Let us try for a solution of the form

$$V(x,y,z) = (\text{function of } x) \times (\text{function of } y) \times (\text{function of } z),$$

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)\,. \tag{98}$$

At the moment we are just trying to get a single solution of the equation. If there is no solution of this kind then we will have to try something else — but of course there will be! Substituting the product form of Eq. (98) into Laplace's equation, we get

$$Y\,Z\,\frac{d^2X}{dx^2} + X\,Z\,\frac{d^2Y}{dy^2} + X\,Y\,\frac{d^2Z}{dz^2} = 0\,. \tag{99}$$

Note that we now have complete differentials (straight d's) because X is a function of only one variable (x), and similarly for Y and Z. Now divide through the equation by the product $V = X\,Y\,Z$ to get

$$\frac{1}{X}\left(\frac{d^2X}{dx^2}\right) + \frac{1}{Y}\left(\frac{d^2Y}{dy^2}\right) + \frac{1}{Z}\left(\frac{d^2Z}{dz^2}\right) = 0\,. \tag{100}$$

Now the first term in Eq. (100) is a function only of x, the second only of y, and the third only of z. **BUT** x, y, and z are <u>independent</u> variables. This means that we could keep y and z fixed and vary just x. In so doing, the second and third terms remain fixed because they only depend upon y and z respectively. Hence the first term must also remain fixed even if x changes. That is, the first term is a constant, as are the second and third. Thus

$$\begin{aligned}\frac{1}{X}\left(\frac{d^2X}{dx^2}\right) &= -\ell^2\,,\\ \frac{1}{Y}\left(\frac{d^2Y}{dy^2}\right) &= -m^2\,,\\ \frac{1}{Z}\left(\frac{d^2Z}{dz^2}\right) &= +n^2\,.\end{aligned} \tag{101}$$

with

$$n^2 = \ell^2 + m^2\,. \tag{102}$$

Note that n^2, ℓ^2 and m^2 are as yet arbitrary constants and could be negative. ℓ, m, n are not necessarily integers.

Have to solve

$$\frac{d^2X}{dx^2} = -\ell^2\,X\,. \tag{103}$$

For real $\ell \neq 0$, this is the simple harmonic oscillator equation

$$X = a_\ell \cos \ell x + b_\ell \sin \ell x\,, \tag{104}$$

where a_ℓ and b_ℓ are arbitrary constants which must be fixed by the boundary conditions. For special case $\ell = 0$, solution simplifies to

$$X = a_0 + b_0\,x\,. \tag{105}$$

If ℓ^2 is negative, put $\ell = il$; the $\cos \ell x$ and $sin\ell x$ become $\cosh lx$ and $i \sinh lx$. Have seen such changes before when studying the damped oscillator in 1B27.

Solutions for Y are similar to those for X, but with m replacing n. For Z have

$$\left(\frac{d^2 Z}{dz^2}\right) = +n^2\, Z\,. \tag{106}$$

Has solutions

$$\begin{aligned} Z = &\; e_n \cosh nz + f_n \sinh nz \qquad (n \neq 0)\,, \\ = &\; e_0 + f_0\, z \qquad (n = 0)\,. \end{aligned} \tag{107}$$

As a consequence, solutions of the separable form <u>do</u> exist. For example, one solution would be with $\ell = 3$, $m = 4$, and $n = 5$.

$$V(x, y, z) = \text{Constant} \times (\sin 3x) \times (\cos 4y) \times (\sinh 5z)$$

is a solution of Laplace's equation, but many more with different values of (ℓ, m, n) exist.

Most general solution is

$$V(x, y, z) = \text{Constant} \times \left\{ \begin{array}{c} \sin \ell x \\ \cos \ell x \end{array} \right\} \times \left\{ \begin{array}{c} \sin my \\ \cos my \end{array} \right\} \times \left\{ \begin{array}{c} \sinh nz \\ \cosh nz \end{array} \right\}$$

with constraint $n^2 = \ell^2 + m^2$.

By the superposition principle, any linear combination of such solutions is also a solution. The most general superposition is

$$\begin{aligned} V(x, y, z) \;\; = \;\; & \sum_{\ell, m} \{a_{\ell m} \cos \ell x + b_{\ell m} \sin \ell x\} \times \{c_{\ell m} \cos my + d_{\ell m} \sin my\} \\ & \times \{e_{\ell m} \cosh nz + f_{\ell m} \sinh nz\}\;. \end{aligned} \tag{108}$$

For any choice of ℓ and m, with $n = \sqrt{\ell^2 + m^2}$, the above product is a solution. Hence the sum is also a solution. Note ℓ and m do not have to be integers and so the above need not be a discrete sum. Also note that if $\ell \to 0$, cosine is replaced by 1 and sine by x.

Imposing boundary conditions

Solution Eq. (108) is quite general, need to relate it potential problem of two parallel plates: have to impose the boundary conditions.

At $z = 0$,

$$V(z=0) = \sum_{\ell m} e_{\ell m} \, \{a_{\ell m} \cos \ell x + b_{\ell m} \sin \ell x\} \times \{c_{\ell m} \cos my + d_{\ell m} \sin my\} = 0$$

for all values of x and y. Hence $e_{\ell m} = 0$ for all ℓ and m. Most general solution simplifies to

$$V(x,y,z) = \sum_{\ell m} \sinh nz \times \{a_{\ell m} \cos \ell x + b_{\ell m} \sin \ell x\} \times \{c_{\ell m} \cos my + d_{\ell m} \sin my\} \ , \quad (109)$$

where coefficient $f_{\ell m}$ has been absorbed into redefined $a_{\ell m}$ and $b_{\ell m}$.
At $z = L$,

$$V(z=L) = \sum_{\ell m} \sinh nL \times \{a_{\ell m} \cos \ell x + b_{\ell m} \sin \ell x\} \times \{c_{\ell m} \cos my + d_{\ell m} \sin my\} = V_0 \ ,$$

for all x and y. Clearly, only solution which gives something independent of x and y is the special case of $\ell = m = n = 0$. Write this explicitly as

$$V(x,y,z) = z\,\{a + bx\}\{c + dy\} \ . \quad (110)$$

At $z = L$,

$$V_0 = L\,\{a + bx\}\{c + dy\}$$

for all (x, y) so that $b = d = 0$ and $ac = V_0/L$. The final solution is, from Eq. (110), the expected

$$V = \frac{V_0 z}{L} \ .$$

Comments

1. Method of solution is *Separation of Variables*: look for a solution which is a product of a function of x times a function of y times a function of z. Reduces problem to that of solving three ordinary differential equations in x, y and z.

2. Have found an infinite number of solutions of the Laplace equation, but have **not** shown that we have found them all.

3. In more complicated examples the ordinary differential equations may be very much harder to solve than the simple oscillator equations here.

4. Unlike the present case, in general you cannot guess the final answer at the start!

3.4 One-dimensional Wave Equation

Seen the wave equation in the first year 1B24 Waves and Optics course. In one dimension, for example a guitar string clamped at $x = 0$ and $x = L$, the displacement $y(x,t)$ obeys

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0\,, \tag{111}$$

where t is the time variable and c the (constant) speed of wave propagation.

Looking for a solution in the form of a product

$$y(x,t) = X(x)\,T(t) \tag{112}$$

leads to

$$T\,\frac{d^2X}{dx^2} - \frac{1}{c^2}\,X\,\frac{d^2T}{dt^2} = 0\,. \tag{113}$$

After dividing out by $y = X\,T$ and taking one term over to the right hand side, we are left with

$$\frac{1}{X}\left(\frac{d^2X}{dx^2}\right) = \frac{1}{c^2T}\left(\frac{d^2T}{dt^2}\right)\,. \tag{114}$$

The left hand side is a function only of x and the right hand side purely of t. Since x and t are independent variables, this means that both sides are equal to a constant, which we shall call $-\omega^2$.

Reduced to solution of two ordinary differential equations

$$\begin{aligned}\left(\frac{d^2X}{dx^2}\right) + \omega^2\,X &= 0\,,\\ \left(\frac{d^2T}{dt^2}\right) + \omega^2 c^2\,T &= 0\,.\end{aligned} \tag{115}$$

Solution of the x equation is

$$X(x) = C\cos\omega x + D\sin\omega x\;,$$

where C and D are arbitrary constants.

Since the boundary conditions are true for all time, we can impose them directly onto $X(x)$. At $x = 0$,

$$X(x=0) = 0 = C\,,\quad\Longrightarrow\quad C = 0\,,$$

whereas at $x = L$,

$$X(x=L) = 0 = D\,sin(\omega L)\quad\Longrightarrow\quad \omega = n\pi/L\,,$$

where $n = 1, 2, 3, \cdots$.

Solving the corresponding "t" equation,

$$\left(\frac{d^2T}{dt^2}\right) + (n\pi c/L)^2\, T = 0\,,$$

gives

$$T = A\,\cos(n\pi ct/L) + B\,\sin(n\pi ct/L)\,,$$

and a total solution of

$$y(x,t) = D\sin(n\pi x/L) \times \{A\,\cos(n\pi ct/L) + B\,\sin(n\pi ct/L)\}\ .$$

This is but one solution and, to get more, we use the superposition principle to find

$$y(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x/L) \times \{A_n\,\cos(n\pi ct/L) + B_n\,\sin(n\pi ct/L)\}\ . \tag{116}$$

Constant D has been absorbed into constants A_n and B_n.

To go further need to impose extra boundary conditions eg shape of string at time $t = 0$. Will look at such problems under Fourier series.

3.5 Laplace's Equation in Spherical Polar Coordinates

Switch to problems with spherical symmetry, important for Quantum Mechanics and atomic physics. If one needs to know the potential due to a charged sphere, it would be perverse to work in Cartesian coordinates. Choose a coordinate system which is appropriate to the boundary conditions to be imposed and, in this case, one should write things down in the spherical polar variables. Last year wrote ∇^2 in plane polar coordinates and it was messy. Unfortunately, in spherical polar coordinates, (r, θ, ϕ), it is even worse! Come to a simpler derivation later in the course. Now

$$\begin{aligned} x =&\ r\sin\theta\cos\phi\,,\\ y =&\ r\sin\theta\sin\phi\,,\\ z =&\ r\cos\theta\,. \end{aligned} \tag{117}$$

The partial derivatives of the Cartesian variables with respect to the polar coordinates are

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi\,,\quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi\,,\quad \frac{\partial z}{\partial r} = \cos\theta\,,$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi\,, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi\,, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta\,.$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi\,, \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi\,, \quad \frac{\partial z}{\partial \phi} = 0\,. \tag{118}$$

Using the chain rule for partial differentiation, we get

$$\begin{aligned}
\frac{\partial}{\partial r} &= \sin\theta\cos\phi\,\frac{\partial}{\partial x} + \sin\theta\sin\phi\,\frac{\partial}{\partial y} + \cos\theta\,\frac{\partial}{\partial z}\,, \\
\frac{\partial}{\partial \theta} &= r\cos\theta\cos\phi\,\frac{\partial}{\partial x} + r\cos\theta\sin\phi\,\frac{\partial}{\partial y} - r\sin\theta\,\frac{\partial}{\partial z}\,, \\
\frac{\partial}{\partial \phi} &= -r\sin\theta\sin\phi\,\frac{\partial}{\partial x} + r\sin\theta\cos\phi\,\frac{\partial}{\partial y}\,.
\end{aligned} \tag{119}$$

These equations can be inverted to find the differentials with respect to Cartesians in terms of those with respect to polar coordinates:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \sin\theta\cos\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\,\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\,\frac{\partial}{\partial \phi}\,, \\
\frac{\partial}{\partial y} &= \sin\theta\sin\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\,\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial \phi}\,, \\
\frac{\partial}{\partial z} &= \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\,.
\end{aligned} \tag{120}$$

The Laplacian operator is the sum of the squares of these three operators,

$$\nabla^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 = \left(\sin\theta\cos\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\,\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\,\frac{\partial}{\partial \phi}\right)^2$$

$$+ \left(\sin\theta\sin\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\,\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial \phi}\right)^2 + \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\right)^2 . \tag{121}$$

Remember that the partial derivative with respect to θ acts for example on the $\sin\theta$ as well. Finally end up with

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\,\frac{\partial V}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2}\,\cot\theta\,\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 V}{\partial \phi^2}\,. \tag{122}$$

This is expression for the Laplacian operator in spherical polar coordinates. Can be written in the slightly more compact form

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial \phi^2}\right)\,. \tag{123}$$

As a check on the form of the operator, consider

$$V = 2x^2 - y^2 - z^2 = r^2\left(2\sin^2\theta\,\cos^2\phi - \sin^2\theta\,\sin^2\phi - \cos^2\theta\right).$$

In Cartesian coordinates, it follows immediately that $\nabla^2 V = 0$. In spherical polar coordinates,

$$
\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial V}{\partial r}\right) &= 6(2\sin^2\theta\,\cos^2\phi - \sin^2\theta\,\sin^2\phi - \cos^2\theta)\,. \\
\frac{\partial V}{\partial \theta} &= r^2(4\sin\theta\cos\theta\cos^2\phi - 2\sin\theta\cos\theta\sin^2\phi + 2\cos\theta\sin\theta)\,. \\
\sin\theta\frac{\partial V}{\partial \theta} &= r^2(4\sin^2\theta\cos\theta\cos^2\phi - 2\sin^2\theta\cos\theta\sin^2\phi + 2\cos\theta\sin^2\theta)\,. \\
\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial V}{\partial\theta}\right) &= 8\cos^2\theta\cos^2\phi - 4\sin^2\theta\cos^2\phi - 4\cos^2\theta\sin^2\phi \\
&\quad +2\sin^2\theta\sin^2\phi - 2\sin^2\theta + 4\cos^2\theta\,. \\
\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} &= 12\sin^2\phi - 6\,.
\end{aligned}
$$

Remarkably enough, the sum of these three terms does in fact vanish!

3.6 Separation of Laplace's equation in Spherical Polar Coordinates

Look for a solution of the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right) = 0 \tag{124}$$

in the form

$$V(r,\,\theta,\,\phi) = R(r)\times\Theta(\theta)\times\Phi(\phi)\,. \tag{125}$$

Involves functions which depend purely upon one variable each, *viz* r, θ and ϕ. Inserting this into Laplace's equation

$$\Theta\,\Phi\,\frac{1}{r^2}\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) + R\,\Phi\,\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + R\,\Theta\,\frac{1}{r^2\sin^2\theta}\left(\frac{d^2\Phi}{d\phi^2}\right) = 0\,.$$

After dividing by $R\,\Theta\,\Phi$ and multiplying by $r^2\sin^2\theta$, find

$$\frac{\sin^2\theta}{R}\,\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) + \frac{1}{\Theta}\,\sin\theta\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\,\left(\frac{d^2\Phi}{d\phi^2}\right) = 0\,.$$

First two terms here depend upon r and θ but third is function purely of azimuthal angle ϕ. Since r, θ and ϕ are independent variables, means that third term must be some constant, denote by $-m^2$. Hence

$$\frac{\partial^2\Phi}{\partial\phi^2} = -m^2\,\Phi\,, \tag{126}$$

which has solutions $e^{\pm im\phi}$ or, alternatively, $\cos m\phi$ and $\sin m\phi$.

As far as DE concerned, m could have any value, even complex. However Physics imposes a fairly general boundary condition. When ϕ increases by 2π, the vector position returns to the same point; expect same physical solution. Thus $\Phi(\phi + 2\pi) = \Phi(\phi)$. Can only be accomplished if m is a real integer. Then $\Phi(\phi)$ is clearly a periodic function.

The remainder of the equation can be manipulated into

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{m^2}{\sin^2\theta} - \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) .$$

Left hand side is function only of r, while right hand side depends only on θ. Means that both sides must be equal to some constant, denote by λ. Results in two ordinary DEs:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda R\,, \tag{127}$$

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + \left(\lambda\sin\theta - \frac{m^2}{\sin\theta}\right)\Theta = 0\,. \tag{128}$$

Now look at the radial equation of Eq. (127), rewritten as

$$r^2\left(\frac{d^2R}{dr^2}\right) + 2r\left(\frac{dR}{dr}\right) - \lambda R = 0\,. \tag{129}$$

This is a special kind of homogeneous equation which is unchanged if the r-variable is scaled as $r \to \alpha\, r$, where α is some constant. Try for a solution of the form $R(r) \sim r^\beta$, since this also stays in same form under the $r \to \alpha r$ scaling. Hence

$$\beta(\beta-1)\,r^\beta + 2\beta\,r^\beta - \lambda\,r^\beta = 0$$

Cancelling out the r^β factor, which cannot vanish, gives $\beta^2 + \beta = \lambda$, has solutions

$$\beta = \left(-1 \pm \sqrt{1+4\lambda}\right)/2\,.$$

Get exactly the same result by trying for the more general series solution. Standard manipulation leads to

$$\sum_{n=0}^{\infty} a_n\left\{(n+k)(n+k+1) - \lambda\right\} r^{n+k} = 0$$

The indicial equation leads to exactly the same result with β replaced by k. For higher values of n have

$$a_n\left\{(n+k)(n+k+1) - \lambda\right\} = a_n\,n(2k+1) = 0\,.$$

But $2k+1 = 2\beta+1 = \pm\sqrt{1+4\lambda}$ doesn't vanish. Hence $a_n = 0$ for $n \geq 1$ and get back to the single-term solution derived above.

To make things look a bit simpler, define eparation constant to be $\lambda \equiv \ell(\ell+1)$, where ℓ is not necessarily an integer. Then

$$\begin{aligned} \beta &= \left(-1 \pm \sqrt{1+4\ell(\ell+1)}\right)/2 \\ &= \ell \ \text{ or } \ -\ell-1\,. \end{aligned}$$

Most general form of the radial solution is

$$R(r) = A\,r^{\ell} + \frac{B}{r^{\ell+1}}\,. \tag{130}$$

In order not to interchange the two solutions, adopt the convention $\ell \geq -\frac{1}{2}$.

Left only with the θ equation which, with new separation constant $\ell(\ell+1)$, becomes

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + \left(\ell(\ell+1)\,\sin\theta - \frac{m^2}{\sin\theta}\right)\,\Theta = 0\,, \tag{131}$$

which does not look very attractive. A little more tractable with the variable $\mu = \cos\theta$ rather than θ. Then $d\mu/d\theta = -\sin\theta$ and

$$\frac{d}{d\theta} = -\sin\theta\,\frac{d}{d\mu} = -\sqrt{1-\mu^2}\,\frac{d}{d\mu}\,.$$

Hence

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\mu^2}\right]\,\Theta = 0\,. \tag{132}$$

This is the famous Legendre differential equation important for quantum mechanics. Legendre discovered his equation when trying to interpret planetary gravitational fields, "Recherches sur la figure des planètes" (1784). This is about 150 years before the discovery of the Schrödinger equation and so you shouldn't blame quantum mechanics for the introduction of Legendre polynomials.