7 Vector Operators

7.1 Operators linear in $\underline{\nabla}$

Last were introduced to the gradient operator. For any differentiable scalar function $f(x, y, z)$, define a vector function

$$grad\, f = \underline{\nabla}\, f = (f_x \underline{\hat{e}}_x + f_y \underline{\hat{e}}_y + f_z \underline{\hat{e}}_z) = \left(\frac{\partial f}{\partial x}\right) \underline{\hat{e}}_x + \left(\frac{\partial f}{\partial y}\right) \underline{\hat{e}}_y + \left(\frac{\partial f}{\partial z}\right) \underline{\hat{e}}_z \,. \tag{313}$$

or as an operator equation

$$\underline{\nabla} = \underline{\hat{e}}_x \frac{\partial}{\partial x} + \underline{\hat{e}}_y \frac{\partial}{\partial y} + \underline{\hat{e}}_z \frac{\partial}{\partial z} \,, \tag{314}$$

where operator $\underline{\nabla}$, pronounced **Del**, acts upon function f.

If the point (x, y, z) is changed by an infinitesimal amount $\underline{dr} = (dx, dy, dz)$

$$df = (f_x \underline{\hat{e}}_x + f_y \underline{\hat{e}}_y + f_z \underline{\hat{e}}_z) \cdot (dx\, \underline{\hat{e}}_x + dy\, \underline{\hat{e}}_y + dz, \underline{\hat{e}}_z) = f_x\, dx + f_y\, dy + f_z\, dz \,, \tag{315}$$

which is expression for total derivative in terms of slopes in three directions.

In case where f is function of $r = \sqrt{x^2 + y^2 + z^2}$ only, then

$$\left(\frac{\partial f}{\partial x}\right) = \frac{df}{dr} \left(\frac{\partial \sqrt{x^2 + y^2 + z^2}}{\partial x}\right) = \frac{x}{r} \frac{df}{dr} \,,$$

wherehave used a chain rule for differentiating. Thus

$$grad\, f = (x,\ y,\ z)\, \frac{1}{r} \frac{df}{dr} = \underline{\hat{e}}_r \frac{df}{dr} \,,$$

where $\underline{\hat{e}}_r$ is a unit vector pointing in the direction of $\underline{r}$.

The divergence operator creates a scalar field from a vector using the scalar product. For a differentiable vector function $\underline{v}(x, y, z)$,

$$div\, \underline{v} = \underline{\nabla} \cdot \underline{v} = \left(\underline{\hat{e}}_x \frac{\partial}{\partial x} + \underline{\hat{e}}_y \frac{\partial}{\partial y} + \underline{\hat{e}}_z \frac{\partial}{\partial z}\right) \cdot \left(v_x \underline{\hat{e}}_x + v_y \underline{\hat{e}}_y + v_z \underline{\hat{e}}_z\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \,. \tag{316}$$

Suppose that $f(x, y, z)$ and $\underline{v}(x, y, z)$ are respectively scalar and vector functions of (x, y, z). Then

$$\underline{\nabla} \cdot (f\, \underline{v}) = \frac{\partial}{\partial x}(f\, v_x) + \frac{\partial}{\partial y}(f\, v_y) + \frac{\partial}{\partial z}(f\, v_z)$$

$$= v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z} + f\, \frac{\partial v_x}{\partial x} + f\, \frac{\partial v_y}{\partial y} + f\, \frac{\partial v_z}{\partial z} = \underline{v} \cdot \underline{\nabla} f + f\, \underline{\nabla} \cdot \underline{v} \,.$$

The vector part of the operator obeys the rules for vectors, whereas the differentiation part obeys the normal rules for differentiation, including that for the derivative of a product.

The curl operator acting on function $\underline{v} = (v_x,\, v_y,\, v_z)$ is defined as

$$curl\, \underline{v} = \underline{\nabla} \times \underline{v}\,. \tag{317}$$

It has components

$$\begin{aligned} (\underline{\nabla} \times \underline{v})_x &= \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\,, \\ (\underline{\nabla} \times \underline{v})_y &= \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\,, \\ (\underline{\nabla} \times \underline{v})_z &= \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\,. \end{aligned} \tag{318}$$

If $curl\, \underline{v} = 0$, the integral

$$W(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \underline{v} \cdot \underline{dr} \tag{319}$$

defines a *unique* function $W(x,y,z)$, whose value does not depend on path of integration.

Examples

1. Saw that $\underline{\nabla}$ obeys simultaneously the rules of differentiation and vector algebra. Also true for *curl.* Thus

$$\underline{\nabla} \times (f\, \underline{v}) = f\, (\underline{\nabla} \times \underline{v}) + (\underline{\nabla}\, f) \times \underline{v}\,,$$

where by convention the differentiation on right hand side only takes place inside the bracket. Simplest proof uses components. Taking just the x-component of the LHS,

$$(\mathrm{LHS})_x = \frac{\partial}{\partial y}(f\, v_z) - \frac{\partial}{\partial z}(f\, v_y)$$

$$= \left(\frac{\partial f}{\partial y}\, v_z - \frac{\partial f}{\partial z}\, v_y\right) + f\, \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) = [(\underline{\nabla}\, f) \times \underline{v}]_x + f\, [\underline{\nabla} \times \underline{v}]_x\,,$$

similarly for the othes.

2. If $\underline{v} = \underline{r}$ and $f = f(r)$, what is $\underline{\nabla} \times (\underline{r}\, f(r))$?

$$\underline{\nabla} \times \underline{r} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \\ \underline{\hat{e}}_x & \underline{\hat{e}}_y & \underline{\hat{e}}_z \end{vmatrix} = 0\,,$$

since the x partial differentiation acts here on y and z, but not x.

Have already shown

$$\underline{\nabla}\, f(r) = \hat{\underline{e}}_r \frac{df}{dr} = \frac{\underline{r}}{r}\frac{df}{dr} \, .$$

Since

$$\underline{r} \times \left(\frac{\underline{r}}{r}\frac{df}{dr} \right) = 0 \, ,$$

means

$$\underline{\nabla} \times (\underline{r}\, f(r)) = 0 \, .$$

7.2 Operators quadratic in $\underline{\nabla}$

The gradient operator takes a scalar into a vector. Acting on the result with the divergence operator gives a scalar again.

$$div\, grad\, f = \underline{\nabla} \cdot (\underline{\nabla}\, f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \, . \tag{320}$$

Resulting operator ∇^2 is called the Laplacian operator, has already been used when discussing the Legendre polynomials. Most of Physics seems to be governed by second order differential equations involving the Laplacian operator, eg Schrödinger eq. describing motion of particle of mass m with energy E in potential $V(r)$;

$$-\frac{\hbar^2}{2m}\nabla^2\, \Psi + V(r)\, \Psi = E\, \Psi \, . \tag{321}$$

In electrostatics, potential $\Phi(\underline{r})$ due to a charge density $\rho(\underline{r})$ satisfies

$$\nabla^2 \Phi = -\frac{1}{4\pi\varepsilon_0}\, \rho \, . \tag{322}$$

There are many more examples.

Other operators which are quadratic in $\underline{\nabla}$, *e.g.*

$$\underline{A} = curl\, grad\, \phi = \underline{\nabla} \times (\underline{\nabla}\, \phi) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ \hat{\underline{e}}_x & \hat{\underline{e}}_y & \hat{\underline{e}}_z \end{vmatrix} \, . \tag{323}$$

This has an x-component of

$$A_x = \frac{\partial}{\partial y}\frac{\partial}{\partial z}\, \phi - \frac{\partial}{\partial z}\frac{\partial}{\partial y}\, \phi = 0 \tag{324}$$

for any reasonable function $\phi(x, y, z)$. Thus

$$\underline{\nabla} \times (\underline{\nabla}\,\phi) = \underline{0}\,. \tag{325}$$

Should recognise this result in the case of an electrostatic field with $\underline{E} = -\underline{\nabla}\,\phi$. The electrostatic field is irrotational

$$\underline{\nabla} \times \underline{E} = \underline{0}\,. \tag{326}$$

Using components can show that

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 0\,. \tag{327}$$

This is another useful result in electromagnetism.

The magnetic induction field $\underline{B}$ is *solenoidal*, *i.e.* $\underline{\nabla} \cdot \underline{B} = 0$. Hence, using Eq. (327), can write

$$\underline{B} = \underline{\nabla} \times \underline{A}\,. \tag{328}$$

In Electromagnetism, $\underline{A}$ is called the magnetic vector potential is needed for quantum description of the interaction of radiation with matter. Magnetic potential appears to be artificial construct which makes $\underline{B}$ automatically solenoidal and not to be a measurable physical quantity. Nevertheless, Aharanov-Bohm effect shows that certain features of the magnetic potential have experimental consequences!

Another quadratic relation is

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla}\,\cdot\,\underline{A}) - \nabla^2 \underline{A}\,. \tag{329}$$

In words this is

curl (curl $\underline{A}$) = grad (div $\underline{A}$) – del squared $\underline{A}$.

This can be proved by writing everything explicitly in terms of components, but there are other methods. For ordinary vectors

$$\underline{d} = \underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}\,\cdot\,\underline{c})\,\underline{b} - (\underline{a}\,\cdot\,\underline{b})\,\underline{c}\,. \tag{330}$$

Trouble with this relation when $\underline{b}$ is also a differential operator, is that cannot change order.. Write in symbolic component form

$$d_i = \sum_j (a_j\, b_i\, c_j - a_j\, b_j\, c_i)\,, \tag{331}$$

without altering order of the vectors. Now put $\underline{a} = \underline{b} = \underline{\nabla}$ and $\underline{c} = \underline{A}$. Since $\underline{a}$ and $\underline{b}$ are now the same operators, it doesn't matter in which order we write them. Hence

$$d_i = \sum_j (\nabla_i \nabla_j A_j - \nabla_j \nabla_j A_i) \, , \tag{332}$$

which is just the component representation of Eq. (329).