MATH 3305 General Relativity Problem sheet 4

Please hand in your solutions Friday, 6 November 2009

Problem 1 (10 points) Compute the Christoffel symbols for the following metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2. \tag{1}$$

Use symmetry $\Gamma^i_{ab} = \Gamma^i_{ba}$ to simplify the calculations.

Problem 2 (30 points) Let us consider the plane $\mathbb{R}^2$ with polar coordinates (r, θ) and metric g_{ij} given by the line element

$$ds^2 = dr^2 + r^2 d\theta^2. \tag{2}$$

Write down the Lagrangian for g_{ij}. To write down the geodesic equation one can start either with the Lagrangian for the metric or the Christoffel symbols for the metric. Which method would you use in this case? Write down the geodesic equation and show that geodesics are straight lines.

Problem 3 (30 points) Solve the geodesic equations of motion for the metric

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2. \tag{3}$$

Interpret your result and comment whether the coordinates are well chosen. Can you suggest a better coordinate system? (Hint: compute $(1/f(t))''$).

Problem 4 (30 points) Show that the Christoffel symbol transforms under general coordinate transformations as follows

$$\widetilde{\Gamma}^a_{bc} = \frac{\partial \widetilde{X}^a}{\partial X^k}\frac{\partial X^l}{\partial \widetilde{X}^b}\frac{\partial X^m}{\partial \widetilde{X}^c}\Gamma^k_{lm} + \frac{\partial^2 X^m}{\partial \widetilde{X}^b \partial \widetilde{X}^c}\frac{\partial \widetilde{X}^a}{\partial X^m}. \tag{4}$$

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The next problem is a bonus problem. Bonus problems can be used to compensate for lost points in this or other exercises.

Bonus Problem (30 points) Suppose $X^1, \ldots, X^n$ and $\widetilde{X}^1, \ldots, \widetilde{X}^n$ are overlapping coordinates on a manifold M.

(i) In X^i and $\widetilde{X}^i$-coordinates, we define functions

$$\begin{aligned} A^i_j(X^1,\dots,X^n) &= \delta^i_j, \\ \widetilde{A}^i_j(\widetilde{X}^1,\dots,\widetilde{X}^n) &= \delta^i_j, \quad i,j=1,\dots,n. \end{aligned}$$

(a) What transformation rules should functions A^i_j and $\widetilde{A}^i_j$ satisfy in order to be components of a $\binom{1}{1}$-tensor?

(b) Show that the above functions satisfy these transformation rules.

(c) Conclude that δ^i_j is a $\binom{1}{1}$-tensor on any manifold M, and describe how this tensor is defined.

(ii) In X^i and $\widetilde{X}^i$-coordinates, we define functions

$$\begin{aligned} B_{ij}(X^1,\dots,X^n) &= \delta_{ij}, \\ \widetilde{B}_{ij}(\widetilde{X}^1,\dots,\widetilde{X}^n) &= \delta_{ij}, \quad i,j=1,\dots,n. \end{aligned}$$

(a) What transformation rules should functions B_{ij} and $\widetilde{B}_{ij}$ satisfy in order to be components of a $\binom{1}{1}$-tensor?

(b) Let us assume that $M=\mathbb{R}^2$, X^i are Cartesian coordinates and $\widetilde{X}^i$ are polar coordinates. What do the transformation rules in (a) look like now?

(c) Conclude that since the transformation rules are not satisfied δ_{ij} is not a tensor on $\mathbb{R}^2$, and hence δ_{ij} do not define a tensor on an arbitrary manifold.

(iii) Let $M=\mathbb{R}^2$. In Cartesian coordinates $(X^1,X^2)=(x,y)$ we define

$$g_{ij}(x,y)=\delta_{ij}.$$

If $(\widetilde{X}^1,\widetilde{X}^2)$ are other local coordinates, we define functions $\widetilde{g}_{ij}(\widetilde{X}^1,\dots,\widetilde{X}^n)$ by

$$\widetilde{g}_{ij}=\frac{\partial X^r}{\partial \widetilde{X}^i}\frac{\partial X^s}{\partial \widetilde{X}^j}g_{rs}.$$

Explain the difference between cases (ii) and (iii). Why is B_{ij} not a tensor, while g_{ij} is a tensor?